

Teleportation stretching for single-mode Gaussian channels

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If a quantum channel is commutable with a certain set of operators, a Choi state of the channel becomes a sufficient resource for any entanglement generation achieved by a single use of the channel with the help of local operation and classical communications (LOCC). This property, called the teleportation stretchable, could be useful to determine an upper bound of a fundamental rate-loss trade-off for optical quantum key distribution. Unfortunately, the known formulation of the teleportation-stretching expansion for lossy Gaussian channels is based on an additional assumption of the Choi state with infinite energy. In this paper, we present an entanglement-assisted LOCC protocol with a bounded energy entangled state, and identify a resource state being sufficient for a general entanglement generation protocol through a single use of lossy Gaussian channels. We also extend this protocol for the cases of general single-mode Gaussian channels. Our results provide a regular path to determine an ultimate limit of quantum communication.

I. INTRODUCTION

Lossy Gaussian channels play a significant role to describe the optical loss and mode mismatches in optical transmission of light fields. Quantifying communication capacities for such channels is a central topic of quantum information science [1–3]. There are many insightful interplay between entanglement and channel capacities in quantum communication under the local operation and classical communication (LOCC) paradigm [4]. An essential question is what is an ultimate limit of sharable entanglement per channel use under arbitrary use of LOCC. Takeoka, Guha, and Wilde introduced the squashed entanglement as an upper bound on the quantum communication capacity of a quantum channel assisted by unlimited forward and backward classical communication, and established a limit of the fundamental rate-loss trade-off in quantum key distribution due to a pure lossy channel [5, 6]. Further, a general upper bound of the rate-loss trade-off on the point-to-point communication assisted with intermediate stations has been formulated in Ref. [7]. Interestingly, there are observations that the bound is not so far from practically achievable rates [8, 9].

In a recent arXiv paper [10] by Pirandola, Laurenza, Ottaviani, and Banchi (PLOB), it is claimed that the tight rate-loss trade-off over a pure lossy channel with transmission η is given by

$$-\log_2(1 - \eta). \quad (1)$$

In order to prove that this is an upper bound, they introduced a method called the teleportation stretching that essentially implies any entanglement generation through a single use of the channel can be accomplished by an LOCC protocol starting with the Choi state of the channel provided the channel fulfills certain conditions [11]. In the case of a finite dimensional systems, the Choi state of a quantum channel \mathcal{E} is normally defined as $\mathcal{I} \otimes \mathcal{E}(\Phi_{\text{MES}})$ with a maximally entangled state Φ_{MES} . As an analogy, PLOB defined a Choi matrix of the pure lossy channel

\mathcal{E}_η acting on an infinite dimensional system as

$$\rho_{\mathcal{E}} := \mathcal{I} \otimes \mathcal{E}_\eta(\Phi_{\text{EPR}}), \quad (2)$$

where they assumed an ideal Einstein-Podolsky-Rosen (EPR) state Φ_{EPR} , which can be formally written as

$$\Phi_{\text{EPR}} = \lim_{\xi \rightarrow 1} |\psi_\xi\rangle\langle\psi_\xi|. \quad (3)$$

Here, the two-mode squeezed state $|\psi_\xi\rangle$ with $\xi \in (0, 1)$ is a realistic EPR state and given by

$$|\psi_\xi\rangle := \sqrt{1 - \xi^2} \sum_{n=0}^{\infty} \xi^n |n\rangle |n\rangle. \quad (4)$$

Due to the teleportation-stretching property, any entanglement generation through a single use of the channel is achieved by an additional LOCC process from the Choi matrix $\rho_{\mathcal{E}}$. Similarly, any entanglement generation through an n -use of the channel can be accomplished by an entanglement-assisted LOCC protocol starting from $\rho_{\mathcal{E}}^{\otimes n}$. Therefore, any entangled state shared through an n -use of the channel with the help of LOCC can be written as

$$\rho_{\mathbf{ab}}^n = \bar{\Lambda}(\rho_{\mathcal{E}}^{\otimes n}), \quad (5)$$

where $\bar{\Lambda}$ stands for an LOCC process, and the subscript of $\rho_{\mathbf{ab}}$ denotes the following condition [10]: The sender Alice prepares an arbitrary quantum state $\rho_{\mathbf{aa}_1\mathbf{a}_2\cdots\mathbf{a}_n}$, and sends an n -mode subsystem $\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n$ to the receiver Bob with the n -channel use; The final subsystem possessed by Bob is specified by the index \mathbf{b} . Given that the final state $\rho_{\mathbf{ab}}^n$ is delivered through an LOCC protocol as in Eq. (5), the monotonicity of entanglement implies that the amount of entanglement shared between the bipartite system \mathbf{ab} is bounded from above as

$$E_R(\rho_{\mathbf{ab}}^n) \leq E_R(\rho_{\mathcal{E}}^{\otimes n}), \quad (6)$$

where E_R is the relative entropy of entanglement (REE), but this relation could hold for any entanglement measure. Due to the additivity of E_R , we may obtain

$$E_R(\rho_{\mathbf{ab}}^n) \leq E_R(\rho_{\mathcal{E}}^{\otimes n}) \leq nE_R(\rho_{\mathcal{E}}). \quad (7)$$

From this relation, the amount of entanglement generated per channel use could be bounded as

$$\frac{E_R(\rho_{\mathbf{ab}}^n)}{n} \leq \frac{E_R(\rho_{\mathcal{E}}^{\otimes n})}{n} \leq E_R(\rho_{\mathcal{E}}) \leq -\log_2(1 - \eta). \quad (8)$$

Notably, the rightmost term $-\log_2(1 - \eta)$ is determined by taking the limit $\xi \rightarrow 1$ for $E_R(\mathcal{I} \otimes \mathcal{E}_\eta(\psi_\xi))$ [10]. Notably, the rightmost term $-\log_2(1 - \eta)$ is determined by taking the limit $\xi \rightarrow 1$ for $E_R(\mathcal{I} \otimes \mathcal{E}_\eta(\psi_\xi))$ [10]. Hence, the RHS of Eq. (3) reads that the limitation $\xi \rightarrow 1$ is taken *eventually*, and until that instance, the ideal EPR state Φ_{EPR} has to be regarded as a normal state since the RHS of Eq. (3) is indeterminate by itself. On the other hand, the central relation in Eq. (5) due to the teleportation-stretching property, is derived under the assumption of the ideal EPR state and thus the limit operation $\xi \rightarrow 1$ is likely to be conducted before Eq. (5). To this end, $\rho_{\mathcal{E}}$ in the RHS of Eq. (5) seems already indeterminate, and this could make the subsequent relations [i.e., Eqs. (6) and (7)] meaningless. This incoherence is obviously coming from the assumption of the ideal EPR state in Eq. (3), and verifying the mathematical validity in heuristic use of such a fictitious state seems difficult.

In this paper, we present a conventional theory of the teleportation-stretching expansion for Gaussian lossy channels, and provide the relation essentially equivalent to Eq. (5) without invoking additional assumptions. We also extend the result for general single-mode Gaussian channels. Thereby, we address the question in a related work [12]: What is a sufficient entangled resource state shared between Alice and Bob for them to prepare an output state $\mathcal{E}(\rho)$ at Bob's station given an arbitrary input state ρ at Alice's station under the restriction of LOCC.

This paper is organized as follows. We start reviewing the teleportation-stretching property for a finite dimensional system, and develop its counterpart for the pure lossy channel in Sec. II. A general result for single-mode Gaussian channels is presented in Sec. III. We conclude this paper in Sec. IV.

II. TELEPORTATION STRETCHING

A. Finite dimension

We start reviewing the teleportation-stretching property for a finite dimensional system associated with the standard quantum teleportation process for a d -dimensional system [13]. Let us write a standard basis $\{|j\rangle\}_{j=0}^{d-1}$. Let $|\phi_0\rangle = \sum_{j=0}^{d-1} |j\rangle |j\rangle$ be a maximally entangled state, and $\{|\phi_k\rangle = (\mathbb{1} \otimes \sigma_k) |\phi_0\rangle\}$ with $k =$

$0, 1, \dots, d^2 - 1$ be a set of Bell states where $\mathbb{1}$ is the identity operator and $\{\sigma_k\}$ represents a set of unitary operators. We define a teleportation set $\{T_k\}$ associated with a quantum teleportation process from system A to system B using the maximally entangled state ϕ_0 between $A'B$ as

$$T_k := {}_{AA'} \langle \phi_k | \phi_0 \rangle_{A'B} = \sigma_k \sum_{j=0}^{d-1} |j\rangle_B \langle j|_A. \quad (9)$$

This implies an arbitrary input state in system A , say ρ_A , is transferred as

$$T_k \rho_A T_k^\dagger = \sigma_k \rho_B \sigma_k^\dagger. \quad (10)$$

Thereby, adjustment of the additional unitary operators $\{\sigma_k\}$ accomplishes the standard teleportation process.

A quantum channel \mathcal{E} is called teleportation stretchable with regard to $\{T_k\}$ [11] if there exists a set of unitary operators $\{U_k\}$ such that

$$\mathcal{E}(T_k \rho T_k^\dagger) = U_k \mathcal{E}(\rho) U_k^\dagger \quad (11)$$

holds for any input state ρ and any k . This property can be equivalently expressed as

$$\mathcal{L}_{U_k^\dagger} \circ \mathcal{E} \circ \mathcal{L}_{T_k}(\rho) = \mathcal{E}(\rho) \quad (12)$$

when we use a shorthand notation:

$$\mathcal{L}_U(\rho) := U \rho U^\dagger. \quad (13)$$

The main implication of Eq. (12) is that the expression in the LHS can be regarded as an LOCC process based on the shared entanglement ϕ_0 . To be specific, knowing the index of Bell measurement k , the owner of system B can obtain the channel output $\mathcal{E}(\rho)$ applying a unitary operation locally, as in the final step of the standard quantum teleportation process. Therefore, any state transmission through the channel \mathcal{E} can be faithfully simulated by an entanglement-assisted LOCC protocol. One can simulate the channel action by locally applying the channel \mathcal{E} after a standard quantum teleportation protocol when the maximally entanglement is shared [12]. In contrast to this fact, the expression in Eq. (12) tells us that such a perfect channel simulation can be done with using a *non-maximally* entangled state corresponding to the Choi state

$$\rho_{\mathcal{E}} := \mathcal{I} \otimes \mathcal{E}(\phi_0), \quad (14)$$

whenever the channel \mathcal{E} fulfills the commutation property described in Eq. (11). An example of teleportation stretchable channels is Pauli channels [11].

Note that the property called the LOCC composability of quantum channels was investigated in Ref. [12]. There, the main question is to identify an entangled state shared between Alice and Bob from which they can prepare an output state $\mathcal{E}(\rho)$ at Bob's station given an arbitrary input state ρ at Alice's station under the use of LOCC.

The relation in Eq. (12) shows that the Choi state $\rho_{\mathcal{E}}$ in Eq. (14) is sufficient to accomplish this task when the channel \mathcal{E} is the teleportation stretchable. On the other hand, it is shown in Ref. [12] that a maximally entangled state whose Schmidt rank is equal to the Schmidt number of the Choi state $\rho_{\mathcal{E}}$ is sufficient for this task.

B. Infinite dimension with a realistic EPR state

In order to consider the teleportation-stretching property in an infinite dimension, an obstacle is that the maximally entangled state ϕ_0 in the limit $d \rightarrow \infty$ becomes indeterminate. Instead, we may use the two-mode squeezed state ψ_{ξ} in Eq. (4) as an effective maximally entangled state as in the continuous-variable (CV) quantum teleportation scheme by Braunstein and Kimble [14]. As well as the form of the maximally entangled state, there should be notable differences in the form of Bell states $\{\phi_k\}$. For a two-mode bosonic field, we may use the simultaneous eigenstates of EPR operators,

$$|\Phi_0\rangle := \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} |x\rangle |x\rangle dx. \quad (15)$$

where $|x\rangle$ represents the eigenstate of the position quadrature \hat{x} , and is normalized by the Dirac- δ function as $\langle x|x'\rangle = \delta(x-x')$. CV-Bell states can be defined as

$$|\Phi_{\beta}\rangle := (\mathbb{1} \otimes D^{\dagger}(\beta)) |\Phi_0\rangle \quad (16)$$

where $D(\beta) = e^{\beta a^{\dagger} - \beta^* a}$ is a displacement unitary operation with the displacement amount $\beta = \frac{1}{\sqrt{2}}(x + ip) \in \mathbb{C}$. A CV-Bell measurement is associated with the following resolution of the identity

$$\int \int |\Phi_{\beta}\rangle \langle \Phi_{\beta}|_{A'B} dx dp = \mathbb{1}_{A'B}. \quad (17)$$

Note that CV-Bell states $\{|\Phi_{\beta}\rangle\}_{\beta \in \mathbb{C}}$ have the normalization of the Dirac- δ function, and should not be regarded as physical state vectors (see, e.g., Chapter 4 of Ref. [15]).

Another essential ingredient for the teleportation-stretching property for lossy Gaussian channels is the property of the displacement covariance. For a pure lossy channel \mathcal{E}_{η} with the transmission $\eta \in (0, 1)$, it holds that

$$\mathcal{E}_{\eta}(D(\beta)\rho D^{\dagger}(\beta)) = D(\sqrt{\eta}\beta)\mathcal{E}_{\eta}(\rho)D^{\dagger}(\sqrt{\eta}\beta). \quad (18)$$

It could be worth noting that a formulation of the teleportation-stretchable property almost parallel to the finite dimensional case is possible if we assume the CV-Bell state Φ_0 is a physical state (See Appendix A).

We consider the stretching process with a two-mode squeezed state ψ_{ξ} defined in Eq. (4). Let us use the Glauber-Sudarshan P representation for an arbitrary input state as

$$\rho = \int_{\alpha \in \mathbb{C}} P_{\alpha} |\alpha\rangle \langle \alpha| d^2\alpha, \quad (19)$$

where $|\alpha\rangle = D(\alpha)|0\rangle$ denotes a coherent state. Suppose that ρ is in system A and ψ_{ξ} is in system $A'B$, and that the CV-Bell measurement will be performed on system AA' , as in the finite dimensional case of Eqs. (9) and (10). From Eqs. (4), (16), (18), and (19), and the property of the displacement operators, a straightforward calculation leads to

$$\begin{aligned} & T_{\beta,\xi} \rho T_{\beta,\xi}^{\dagger} \\ &:= \langle \Phi_{\beta} | (\rho \otimes |\psi_{\xi}\rangle \langle \psi_{\xi}|)_{AA'B} | \Phi_{\beta} \rangle_{AA'} \\ &= \int P_{\alpha} {}_{AA'} \langle \Phi_{\beta} | \alpha \rangle_A (|\psi_{\xi}\rangle \langle \psi_{\xi}|)_{A'B} {}_A \langle \alpha | \Phi_{\beta} \rangle_{AA'} d^2\alpha \\ &= \frac{1}{2\pi} \int P_{\alpha} \langle \alpha^* | D_{A'}(\beta) |\psi_{\xi}\rangle \langle \psi_{\xi}| D_{A'}^{\dagger}(\beta) | \alpha^* \rangle_{A'} d^2\alpha \\ &= \frac{1}{2\pi} \int P_{\alpha} \langle \alpha^* + \beta^* | \psi_{\xi} \rangle \langle \psi_{\xi} | \alpha^* + \beta^* \rangle d^2\alpha \\ &= \frac{1-\xi^2}{2\pi} \int P_{\alpha} e^{-(1-\xi^2)|\alpha+\beta|^2} |\xi(\alpha+\beta)\rangle \langle \xi(\alpha+\beta)| d^2\alpha \\ &= \frac{1-\xi^2}{2\pi} D(\xi\beta) \left(\int P_{\alpha} e^{-(1-\xi^2)|\alpha+\beta|^2} |\xi\alpha\rangle \langle \xi\alpha| d^2\alpha \right) D^{\dagger}(\xi\beta) \\ &= D(\xi\beta) (\mathcal{E}_{\xi^2}(\tilde{\rho}_{\beta})) D^{\dagger}(\xi\beta), \end{aligned} \quad (20)$$

where, in the final step, we used the property of the lossy channel, that is, $\mathcal{E}_{\xi^2}(|\alpha\rangle \langle \alpha|) = |\xi\alpha\rangle \langle \xi\alpha|$ holds for any coherent states $|\alpha\rangle$ with $\alpha \in \mathbb{C}$, and defined

$$\tilde{\rho}_{\beta} := \frac{1-\xi^2}{2\pi} \int P_{\alpha} e^{-(1-\xi^2)|\alpha+\beta|^2} |\alpha\rangle \langle \alpha| d^2\alpha. \quad (21)$$

For later use we note the following relation:

$$\int \int \tilde{\rho}_{\beta} dx dp = 2 \int \tilde{\rho}_{\beta} d^2\beta = \int P_{\alpha} |\alpha\rangle \langle \alpha| d^2\alpha = \rho. \quad (22)$$

With the help of Eq. (18) and another property of the lossy channel $\mathcal{E}_{\eta} \circ \mathcal{E}_{\eta'} = \mathcal{E}_{\eta\eta'}$, Eq. (20) implies

$$\begin{aligned} \mathcal{E}_{\eta} \left(T_{\beta,\xi} \rho T_{\beta,\xi}^{\dagger} \right) &= \mathcal{E}_{\eta} \left(D(\xi\beta) (\mathcal{E}_{\xi^2}(\tilde{\rho}_{\beta})) D^{\dagger}(\xi\beta) \right) \\ &= D(\sqrt{\eta}\xi\beta) (\mathcal{E}_{\eta}(\mathcal{E}_{\xi^2}(\tilde{\rho}_{\beta}))) D^{\dagger}(\sqrt{\eta}\xi\beta) \\ &= D(\sqrt{\eta}\xi\beta) (\mathcal{E}_{\eta\xi^2}(\tilde{\rho}_{\beta})) D^{\dagger}(\sqrt{\eta}\xi\beta). \end{aligned} \quad (23)$$

This relation is slightly different from the formula in Eq. (11), and the teleportation-stretching property cannot hold for the realistic CV-teleportation set $\{T_{\beta,\xi}\}$. However, we can show a modified version of such a property sufficient to simulate the channel action. We may say that the pure lossy channel is *effectively teleportation-stretchable* regarding the following relation:

$$\int D^{\dagger}(\sqrt{\eta}\xi\beta) \mathcal{E}_{\eta} \left(T_{\beta,\xi} \rho T_{\beta,\xi}^{\dagger} \right) D(\sqrt{\eta}\xi\beta) dx dp = \mathcal{E}_{\eta\xi^2}(\rho). \quad (24)$$

This relation can be readily obtained by an application of $\mathcal{L}_{D^{\dagger}(\sqrt{\eta}\xi\beta)}$ and integration over β for both sides of Eq. (23) with the help of Eq. (22). As a counterpart of

the expression in Eq. (12), the following representation would be helpful:

$$\int \left(\mathcal{L}_{D^\dagger(\sqrt{\eta}\xi\beta)} \circ \mathcal{E}_\eta \circ \mathcal{L}_{T_{\beta,\xi}}(\rho) \right) dxdp = \mathcal{E}_{\eta\xi^2}(\rho). \quad (25)$$

In both the representations in Eqs. (24) and (25), the expression of the LHSs is regarded as an LOCC process based on the realistic EPR source ψ_ξ and a single use of the lossy channel \mathcal{E}_η . Therefore, the central implication of our result is that any transmission through a lossy channel \mathcal{E}_η can be simulated by an entanglement-assisted LOCC with the resource state generated through another lossy channel $\mathcal{E}_{\eta'}$ and a two-mode squeezed state ψ_ξ . Let us restate this relation as a theorem:

Theorem.—Let \mathcal{E}_η be a pure lossy channel with transmission $\eta \in (0, 1)$. For any two-mode state ρ_{AB} , there exists another pure lossy channel $\mathcal{E}_{\eta'}$ with $\eta' \geq \eta$, a two-mode squeezed state ψ_ξ , and an LOCC process Λ , such that

$$\mathcal{I} \otimes \mathcal{E}_\eta(\rho_{AB}) = \Lambda(\rho_{\mathcal{E}'}), \quad (26)$$

where an effective Choi state $\rho_{\mathcal{E}'}$ is defined as

$$\rho_{\mathcal{E}'} := \mathcal{I} \otimes \mathcal{E}_{\eta'}(\psi_\xi). \quad (27)$$

The LOCC process can be represented by the effective teleportation-stretchable condition of Eq. (24) (or equivalently Eq. (25)). The parameters (η', ξ) can be chosen so as to satisfy $\eta' = \eta/\xi^2$.

Proof.—Obvious from the effective teleportation-stretchable property in Eq. (24) [or equivalently, Eq. (25)]. ■

Physically, the use of the realistic EPR state ψ_ξ implies a teleportation resource of a non-maximally entangled state. Then, the resultant teleportation cannot be perfect, and a part of the imperfection is equivalently induced by a lossy channel as in Eq. (20). Thereby, compensation of the gain seems to be essential if one insists that the unit-gain condition is crucial, although the effect of loss is inherently irreversible [16, 17]. In contrast, our goal here is to simulate the lossy channel \mathcal{E}_η , and we thus rather make use of the imperfection as a natural tool to execute the precise simulation.

C. Step for a regular proof

Now we can establish a regular step to find the bound of Eq. (1) claimed in Ref. [10] as follows. Since, any two-mode state transmission can be simulated by an entanglement-assisted LOCC protocol as in the RHS of Eq. (26), any state transmission plus LOCC operation can be also simulated by an entanglement-assisted LOCC protocol from the Choi state $\rho_{\mathcal{E}'}$. We can apply this transmission-plus-LOCC process for entanglement generation based on the transmission of an arbitrary n modes. Therefore, any entanglement generated by an n -use of

channel and LOCC can be achieved by an LOCC protocol when $\rho_{\mathcal{E}'}^{\otimes n}$ is shared between the bipartite system **ab**. This establishes a finite-energy counterpart of Eq. (5) as

$$\rho_{\mathbf{ab}}^n = \bar{\Lambda}(\rho_{\mathcal{E}'}^{\otimes n}). \quad (28)$$

Hence, we can find an upper bound of entanglement due to the LOCC monotonicity without additional assumptions. For REE, it holds that

$$E_R(\rho_{\mathbf{ab}}^n) \leq E_R(\rho_{\mathcal{E}'}^{\otimes n}). \quad (29)$$

We can repeat the flow of Eqs. (6) and (7), and eventually take the limit $\xi \rightarrow 1$ so as to reach the upper bound in Eq. (1). As a consequence, we can safely remove the assumption of the ideal EPR resource Φ_{EPR} .

III. COMPOSING SINGLE-MODE GAUSSIAN CHANNELS VIA ENTANGLEMENT-ASSISTED LOCC PROTOCOLS

We can expand our stretching property for general single-mode Gaussian channels, and address the problem how to simulate single-mode Gaussian channels via entanglement-assisted LOCC protocols in the sense of Ref. [12]. Due to the unitary-equivalent decomposition of single-mode Gaussian channels [18, 19], it is sufficient to consider a couple of specific channels described by the unitary-equivalent standard forms. There are essentially two standard channels (standard forms) which require entanglement for this LOCC task. This is because except for them the standard channels belong to entanglement breaking channels, and can be simulated solely by LOCC.

One of the relevant standard channels is the phase-insensitive channel, which is further divided into the phase-insensitive lossy channel \mathcal{E}_η^N , and phase-insensitive amplification channel \mathcal{A}_κ^N . Here, $\kappa \geq 1$ is the amplification gain and $N \geq 0$ stands for the excess noise (See Appendix B for detail). Note that the pure lossy channel can be written as $\mathcal{E}_\eta = \mathcal{E}_\eta^0$, and the amplification channel with no excess noise \mathcal{A}_κ^0 is often referred to as the quantum-limited phase-insensitive amplifier. Essentially the same displacement-covariance property as in Eq. (18) holds for both \mathcal{E}_η^N and \mathcal{A}_κ^N . To be specific, \mathcal{A}_κ^N satisfies $\mathcal{A}_\kappa^N(D(\beta)\rho D^\dagger(\beta)) = D(\sqrt{\kappa}\beta)\mathcal{A}_\kappa^N(\rho)D^\dagger(\sqrt{\kappa}\beta)$.

Using \mathcal{E}_η^N instead of \mathcal{E}_η in Eq. (23) and following the steps toward Eq. (24) with the help of Eq. (B6), we obtain the effective stretching property for general lossy channels

$$\int \left(\mathcal{L}_{D^\dagger(\sqrt{\eta}\xi\beta)} \circ \mathcal{E}_\eta^N \circ \mathcal{L}_{T_{\beta,\xi}}(\rho) \right) dxdp = \mathcal{E}_{\eta\xi^2}^N(\rho). \quad (30)$$

With the replacement $\eta\xi^2 \rightarrow \eta$, we can write

$$\int \left(\mathcal{L}_{D^\dagger(\sqrt{\eta}\beta)} \circ \mathcal{E}_{\eta\xi^{-2}}^N \circ \mathcal{L}_{T_{\beta,\xi}}(\rho) \right) dxdp = \mathcal{E}_\eta^N(\rho), \quad (31)$$

where we require $\xi^2 > \eta$ in order that $\mathcal{E}_{\eta\xi^{-2}}^N$ is a lossy channel. Essentially the same procedure for the amplification channel \mathcal{A}_κ^N with the help of Eq. (B8) and some refinement yields

$$\int \left(\mathcal{L}_{D^\dagger(\sqrt{\kappa}\beta)} \circ \mathcal{A}_{\kappa\xi^{-2}}^N \circ \mathcal{L}_{T_{\beta,\xi}}(\rho) \right) dxdp = \mathcal{A}_\kappa^{\tilde{N}}(\rho), \quad (32)$$

where

$$\tilde{N} = N + \kappa(1 - \xi^2)/\xi^2. \quad (33)$$

Similar to the case of Eq. (25), the LHS expressions of Eqs (31) and (32) are regarded as entanglement-assisted LOCC protocols which precisely reproduce the channel action of the RHS terms. Therefore, we can identify the entangled state $\mathcal{I} \otimes \mathcal{E}_{\eta\xi^{-2}}^N(\psi_\xi)$ as a sufficient resource to simulate the channel \mathcal{E}_η^N , and the entangled state $\mathcal{I} \otimes \mathcal{A}_{\kappa\xi^{-2}}^N(\psi_\xi)$ as a sufficient resource to simulate the channel \mathcal{A}_κ^N .

The other relevant standard channel is the so-called additive noise channel \mathcal{E}_{ANC} , which becomes a phase-insensitive channel when coupled with any finite loss as is shown in Theorem 4 of Ref. [20]. Therefore, it can be simulated as a phase-sensitive channel, practically. Since, \mathcal{E}_{ANC} is equivalent to the identity operation except it adds a finite noise on a single quadrature variable, an exact LOCC composition seems necessitate accomplishing the perfect teleportation as the identity operation. Formally, we can set $\kappa = 1$ and $\xi \rightarrow 1$ in Eq. (32) to achieve the perfect teleportation, but we must accept a finite error since we assume $\xi < 1$.

Notably, the quantum limited amplifier \mathcal{A}_κ^0 has the same constraint that the LOCC simulation comes with a finite error. This can be seen from the fact $\tilde{N} - N = 0$ holds only for $\xi = 1$ in Eq. (33). In sharp contrast, the lossy channel \mathcal{E}_η^N can be perfectly simulated without concerning the condition $\xi \rightarrow 1$, all over the parameter space $N \geq 0$ and $\eta \in (0, 1)$. In this sense, \mathcal{E}_η^N seems exceptional, although it remains the possibility that \mathcal{A}_κ^0 or \mathcal{E}_{ANC} can be perfectly simulated by a different LOCC protocol with the help of a different type of entangled resource.

IV. CONCLUSION AND REMARKS

We have pointed out an illness of the assumption used in the proof of the tight upper bound of the ultimate communication rate over a lossy channel in Ref. [10]. We have developed an effective formula of the teleportation-stretching property for lossy Gaussian channels. This enables us to retrieve a sequence of central inequalities appeared in the step of the proof in Ref. [10] without introducing additional assumptions. We have also extended the resultant formula for general single-mode Gaussian channels. This extension shows how to simulate single-mode Gaussian channels via entanglement-assisted

LOCC protocol in the sense of Ref. [12]. Our result would provide a regular path to establish the definitive form of the fundamental capacities of optical quantum communication, and be widely useful to comprehend fundamental properties between quantum entanglement and quantum channels. It remains open to what extent one can generalize the present results for multi-mode Gaussian channels.

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Appendix A: Non-physical algebraic relations

If we assume the CV-Bell state $|\Phi_0\rangle$ is a physical state, the teleportation set could be defined as

$$T_\beta := {}_{AA'}\langle\Phi_\beta|\Phi_0\rangle_{A'B} = \frac{1}{2\pi} D_B(\beta) \int |x\rangle_B \langle x|_A dx. \quad (A1)$$

The action of T_β implies a state transfer from system A to system B with an additional displacement, namely,

$$T_\beta |\varphi\rangle_A = \frac{1}{2\pi} D_B(\beta) |\varphi\rangle_B. \quad (A2)$$

This superficially shows that an adjustment of the local displacement $D(\beta)$ at system B accomplishes the quantum teleportation process.

From Eqs. (A1) and (18), one can write

$$\mathcal{L}_{D^\dagger(\sqrt{\eta}\beta)} \circ \mathcal{E}_\eta \circ \mathcal{L}_{T_\beta}(\rho) = \frac{1}{(2\pi)^2} \mathcal{E}_\eta(\rho). \quad (A3)$$

As an analogy with the finite dimension case of Eq. (12), one may interpret the LHS of Eq. (A3) as an LOCC protocol associated with the CV-Bell state Φ_0 . However, this is no more than an interpretation because Eq. (A3) over the integration of β does not give a properly normalized output density operator.

Note that the ideal EPR state Φ_{EPR} in Eq. (3) has a different normalization compared with the CV-Bell state Φ_0 . In fact, its representation in the Fock basis is given by

$$|\Phi_0\rangle = \left(\sum_{n=0}^{\infty} |n\rangle \langle n| \otimes \mathbb{1} \right) |\Phi_0\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} |n\rangle |n\rangle. \quad (A4)$$

This expression comes from Eq. (15) together with the fact that the wave function of the Fock states can be real, i.e., $\langle n|x\rangle = \langle x|n\rangle$.

Appendix B: Single-mode Gaussian channels

We summarize basic description for single-mode Gaussian channels, and useful relations for derivation of Eqs. (31) and (32).

Any single-mode Gaussian channel can be described by a triplet of 2-by-2 matrix (K, m, α) [19]. The pair (K, α) and a 2-by-2 covariance matrix of physical states γ respectively fulfill the physical conditions

$$\alpha \geq \frac{i}{2}(\sigma - K^T \sigma K), \quad \gamma \geq \frac{i}{2}\sigma. \quad (\text{B1})$$

Here, and what follows we may use the following symbols

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{1}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B2})$$

The channel action transforms the covariant matrix γ and first moment d of physical states as

$$\gamma' = K^T \gamma K + \alpha, \quad d' = K^T d + m. \quad (\text{B3})$$

The last equation suggests m is a state-independent constant shift, and we may set $m = 0$ without notification when the constant displacement makes no physical difference. We can see that any composition of two Gaussian channels \mathcal{E}_1 and \mathcal{E}_2 yields another Gaussian channel $\mathcal{E}_{12} = \mathcal{E}_2 \circ \mathcal{E}_1$, and the triplet of \mathcal{E}_{12} fulfills

$$\begin{aligned} K_{12} &= K_1 K_2, \\ m_{12} &= K_2^T m_1 + m_2, \\ \alpha_{12} &= K_2^T \alpha_1 K_2 + \alpha_2. \end{aligned} \quad (\text{B4})$$

The lossy channel \mathcal{E}_η^N is associated with the triplet $(K, 0, \alpha)$ with

$$K = \sqrt{\eta} \mathbb{1}_2, \quad \alpha = \left(\frac{1-\eta}{2} + N \right) \mathbb{1}_2. \quad (\text{B5})$$

The pure lossy channel is the special case of \mathcal{E}_η^N with $N = 0$, i.e., $\mathcal{E}_\eta := \mathcal{E}_\eta^0$. Using Eqs. (B4) and (B5), we can obtain the combination relation of the pure lossy channel $\mathcal{E}_\eta \circ \mathcal{E}_{\eta'} = \mathcal{E}_{\eta\eta'}$, which is used in the derivation of Eq. (23). Similarly, a straightforward calculation leads to a general formula useful to prove the relation of Eq. (31):

$$\mathcal{E}_\eta^N \circ \mathcal{E}_{\eta'}^{N'} = \mathcal{E}_{\eta\eta'}^{N+\eta N'}. \quad (\text{B6})$$

The amplification channel \mathcal{A}_κ^N is described by the triplet $(K, 0, \alpha)$ with

$$K = \sqrt{\kappa} \mathbb{1}_2, \quad \alpha = \left(\frac{\kappa-1}{2} + N \right) \mathbb{1}_2. \quad (\text{B7})$$

For a possible composition of the lossy channel and the amplification channel, we can show from Eqs. (B4), (B5), and (B7) that the relation

$$\mathcal{A}_\kappa^N \circ \mathcal{E}_{\xi^2} = \mathcal{A}_{\kappa\xi^2}^{N+\kappa(1-\xi^2)} \quad (\text{B8})$$

holds for $\kappa\xi^2 \geq 1$. A refinement of this relation yields

$$\mathcal{A}_{\kappa\xi^{-2}}^N \circ \mathcal{E}_{\xi^2} = \mathcal{A}_\kappa^{N+\kappa\left(\frac{1-\xi^2}{\xi^2}\right)}, \quad (\text{B9})$$

where we assume $\kappa \geq 1$. The relation in Eq. (B8) or its refined form in Eq. (B9) is essential for the derivation of Eq. (32).

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